Mixed Partials – Young's Theorem

This document contains a proof of the equality of mixed partials under a natural assumption. The theorem is due to W. H. Young [1], but his proof is hard to follow. I hope this proof is easy to follow. It is essentially due to Dieudonne. There is a nice exposition in Pugh's text. This exposition is that proof, with perhaps simpler notation.

Theorem 1. *Suppose $f(x,y)$ is defined in a neighborhood of a point (a,b). Suppose the partial derivatives f_x, f_y are defined in a neighborhood of (a,b) and are differentiable at (a,b). (In particular this implies that f_x, f_y are continuous at (a,b), but it is not assumed that their derivatives exist anywhere other than at (a,b).) A short statement of the assumption is that $Df = [f_x, f_y]$ is differentiable at (a,b). This is sometimes stated as f is twice differentiable at (a,b). Then*

$$(f_x)_y(a,b) = (f_y)_x(a,b),$$

sometimes stated as

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Proof. Consider the function

$$\Delta(t) = [f(a+t,b+t) - f(a+t,b)] - [f(a,b+t) - f(a,b)].$$

If we let $g(s) = f(a+s,b+t) - f(a+s,b)$

$$\Delta(t) = g(t) - g(0).$$

By the mean value theorem

$$\begin{aligned}
\Delta(t) &= g(t) - g(0) = tg'(\xi) && (1)\\
&= [f_x(a+\xi,b+t) - f_x(a+\xi,b)]t && (2)\\
&= [f_x(a,b) + (f_x)_x(a,b)\xi + (f_x)_y(a,b)t + p_1(t)\xi + p_2(t)t]t && (3)\\
&\quad - [f_x(a,b) + (f_x)_x(a,b)\xi + q_1(t)\xi]t && (4)\\
&= (f_x)_y(a,b)t^2 + [p_1(t)\xi + q_1(t)\xi + p_2(t)t]t. && (5)
\end{aligned}$$

Now divide by t^2 and remember that $|\xi| < |t|$ to get

$$\frac{\Delta(t)}{t^2} = (f_x)_y(a,b) + [p_1(t)\xi + q_1(t)\xi + p_2(t)t]/t.$$

The last term goes to 0 as $t \to 0$. Hence

$$(f_x)_y(a,b) = \lim_{t\to 0} \frac{\Delta(t)}{t^2}.$$

The argument is symmetric in x and y, so

$$(f_x)_y(a,b) = \lim_{t\to 0} \frac{\Delta(t)}{t^2} = (f_y)_x(a,b).$$

□

Neat, huh!

[1]. Young, W. H., "On the Conditions for the Reversibility of the Order of Partial Differentiation", Proceedings Royale Society of Edinburgh, v. 29 (1908-09), p. 136-164.